

Fundamentals of Quantum Gravity*

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Abstract

The outline of a recent approach to quantum gravity is presented. Novel ingredients include: (1) Affine kinematical variables; (2) Affine coherent states; (3) Projection operator approach toward quantum constraints; (4) Continuous-time regularized functional integral representation without/with constraints; and (5) Hard core picture of nonrenormalizability. The “diagonal representation” for operator representations, introduced by Sudarshan into quantum optics, arises naturally within this program.

Introduction

Nearly 40 years ago, George Sudarshan and the present author published our book “Fundamentals of Quantum Optics”, [1] and it is noteworthy that this book has been recently reprinted by Dover [2]. The title of the present paper is meant to honor the title of our earlier joint work, but in fact, it is also meant in a literal sense as well in that the approach to be outlined in this paper does constitute, in the author’s opinion, a fundamental view of how quantum gravity can be approached. It is important at the outset to remark that what is presented here is not string theory nor is it loop

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quantum gravity, the two most commonly studied approaches to quantum gravity. In the author's judgment, the present approach, known as Affine Quantum Gravity, is more natural than either of the traditional views and is closer to classical (Einstein) gravity as well. General references for this paper are [3, 4, 5].

The paper is divided into several sections each representing a fundamental building block in the edifice we hope to construct. The basic building blocks are designed to address essential components of any natural approach to a quantum field theory. Section 1 addresses the question of just what constitutes the proper choice of a fundamental set of kinematical, phase space, field variables.

In Sec. 2 we first observe that the gravitational field theory is a special theory in that all its dynamical content is enforced by constraints. Since quantization normally requires a phase space geometry with a high degree of symmetry, it follows that it is generally prudent to quantize first and reduce second because if one reduces first, there is generally no guarantee that the reduced classical phase space still has sufficient symmetry to ensure an ambiguity-free quantization. Consequently, in Sec. 2 we choose a representation of the field operators (from among uncountably many inequivalent, irreducible choices!) before we have imposed any of the constraints.

Section 3 takes up the question of the quantum constraints. It is characteristic of gravity that it is classically an open first-class constraint system, meaning that it fulfills a Lie-algebra like set of mutual Poisson brackets among the constraints save for the fact that instead of structure constants there are structure functions of the phase space variables. When quantized, these structure functions become operators which, in the case of gravity, do not commute with the constraints and thereby lead to a set of quantum constraints that are partially second class in nature. Normally, such an anomalous behavior requires the introduction of unphysical, auxiliary variables, if it can be treated at all. However, there is a relatively new approach to deal with operator constraints that treats both first- and second-class constraints in exactly the same way, and as a consequence, this method, which is explained in Sec. 3, seems to be ideal to analyze the gravitational field.

Functional integral methods are valuable as guides in quantization. Sometimes, one is fortunate to actually evaluate the integral, but even when that is not possible, the form of the integral itself can sometimes be used to draw useful qualitative conclusions. Therefore, it is important to observe that

the present formulation of quantum gravity admits a reasonably well-defined functional integral, both in the initial case without imposition of any constraints, as well as in the case in which the constraints are introduced; this general subject is discussed in Sec. 4.

Finally, we face the conventional wisdom that gravity is a perturbatively nonrenormalizable theory. To deal with this situation we recall the hard core theory of nonrenormalizable theories in general. This theory asserts that nonrenormalizable quantum field theories behave as they do because, from a functional integral point of view, the nonlinear interaction term acts as a hard core projecting out field distributions that would otherwise have been allowed by the free theory alone. As a consequence, as with any hard-core interaction, the interacting theory does *not* reduce to the free theory (in the sense of Green's function convergence) as the coupling constant vanishes, and thus the use of regularized perturbation about the free theory to suggest counterterms to the quantum theory is inappropriate. In Sec. 5, we outline the hard-core theory for quantum gravity, which although we do not have explicit control of such hard cores is nonetheless highly suggestive.

1 Affine Kinematical Variables

Metric positivity

An essential property of affine quantum gravity is the strict positivity of the spatial metric. For the classical metric, this property means that for any nonvanishing set $\{u^a\}$ of real numbers and any nonvanishing, nonnegative test function, $f(x) \geq 0$, that

$$\int f(x) u^a g_{ab}(x) u^b d^3x > 0 , \quad (1)$$

where $1 \leq a, b \leq 3$. We also insist that this inequality holds when the classical metric field $g_{ab}(x)$ is replaced with the 3×3 operator metric field $\hat{g}_{ab}(x)$.

Affine commutation relations

The canonical commutation relations are not compatible with the requirement of metric positivity since the canonical momentum acts to translate

the spectrum of the metric tensor, and such a translation is incompatible with metric positivity. Thus it is necessary to find a suitable but distinctly alternative set of commutation relations. A suitable alternative that has the virtue of preserving the spectrum of a positive definite metric operator is readily available.

The initial step involves replacing the classical ADM canonical momentum $\pi^{ab}(x)$ with the classical mixed-index momentum $\pi_b^a(x) \equiv \pi^{ac}(x)g_{cb}(x)$. We refer to $\pi_b^a(x)$ as the “momentric” tensor being a combination of the canonical *momentum* and the canonical *metric*. Besides the metric being promoted to an operator $\hat{g}_{ab}(x)$, we also promote the classical momentric tensor to an operator field $\hat{\pi}_b^a(x)$; this pair of operators form the basic kinematical affine operator fields, and all operators of interest are given as functions of this fundamental pair. The basic kinematical operators are chosen so that they satisfy the following set of *affine commutation relations* (in units where $\hbar = 1$, which are normally used throughout):

$$\begin{aligned} [\hat{\pi}_b^a(x), \hat{\pi}_d^c(y)] &= \frac{1}{2}i[\delta_b^c\hat{\pi}_d^a(x) - \delta_d^a\hat{\pi}_b^c(x)]\delta(x, y) , \\ [\hat{g}_{ab}(x), \hat{\pi}_d^c(y)] &= \frac{1}{2}i[\delta_a^c\hat{g}_{bd}(x) + \delta_b^c\hat{g}_{ad}(x)]\delta(x, y) , \\ [\hat{g}_{ab}(x), \hat{g}_{cd}(y)] &= 0 . \end{aligned} \quad (2)$$

These commutation relations arise as the transcription into operators of equivalent Poisson brackets for the corresponding classical fields, namely, the spatial metric $g_{ab}(x)$ and the momentric field $\pi_d^c(x) \equiv \pi^{cb}(x)g_{bd}(x)$, along with the usual Poisson brackets between the canonical metric field $g_{ab}(x)$ and the canonical momentum field $\pi^{cd}(x)$.

The virtue of the affine variables and their associated commutation relations is evident in the relation

$$e^{i\int \gamma_b^a(y)\hat{\pi}_a^b(y) d^3y} \hat{g}_{cd}(x) e^{-i\int \gamma_b^a(y)\hat{\pi}_b^a(y) d^3y} = \{e^{\gamma(x)/2}\}_c^e \hat{g}_{ef}(x) \{e^{\gamma(x)^T/2}\}_d^f , \quad (3)$$

where $\gamma^T(x)$ denotes the transpose of the matrix $\gamma(x)$. This algebraic relation confirms that suitable transformations by the momentric field preserve metric positivity.

2 Affine Coherent States

It is noteworthy that the algebra generated by \hat{g}_{ab} and $\hat{\pi}_b^a$ as represented by (2) closes. These operators form the generators of the *affine group* whose

elements may be defined by

$$U[\pi, \gamma] \equiv e^{i \int \pi^{ab}(y) \hat{g}_{ab}(y) d^3y} e^{-i \int \gamma_b^a(y) \hat{\pi}_a^b(y) d^3y} , \quad (4)$$

e.g., for all real, smooth c -number functions π^{ab} and γ_b^a of compact support. Since we assume that the smeared fields \hat{g}_{ab} and $\hat{\pi}_b^a$ are self-adjoint operators, it follows that $U[\pi, \gamma]$ are unitary operators for all π and γ , and moreover, these unitary operators are strongly continuous in the label fields π and γ .

To define a representation of the basic operators it suffices to choose a fiducial vector and thereby to introduce a set of affine coherent states, i.e., coherent states formed with the help of the affine group. We choose $|\eta\rangle$ as a normalized fiducial vector in the original Hilbert space \mathfrak{H} , and we consider a set of unit vectors each of which is given by

$$|\pi, \gamma\rangle \equiv e^{i \int \pi^{ab}(x) \hat{g}_{ab}(x) d^3x} e^{-i \int \gamma_c^d(x) \hat{\pi}_d^c(x) d^3x} |\eta\rangle . \quad (5)$$

As π and γ range over the space of smooth functions of compact support, such vectors form the desired set of coherent states. The specific representation of the kinematical operators is fixed once the vector $|\eta\rangle$ has been chosen. As minimum requirements on $|\eta\rangle$ we impose

$$\langle \eta | \hat{\pi}_b^a(x) | \eta \rangle = 0 , \quad (6)$$

$$\langle \eta | \hat{g}_{ab}(x) | \eta \rangle = \tilde{g}_{ab}(x) , \quad (7)$$

where $\tilde{g}_{ab}(x)$ is a metric that determines the topology of the underlying space-like surface. As algebraic consequences of these conditions, it follows that

$$\langle \pi, \gamma | \hat{g}_{ab}(x) | \pi, \gamma \rangle = \{e^{\gamma(x)/2}\}_a^c \tilde{g}_{cd}(x) \{e^{\gamma(x)^T/2}\}_b^d \equiv g_{ab}(x) , \quad (8)$$

$$\langle \pi, \gamma | \hat{\pi}_c^a(x) | \pi, \gamma \rangle = \pi^{ab}(x) g_{bc}(x) \equiv \pi_c^a(x) . \quad (9)$$

These expectations are not gauge invariant, nor should they be, since they are taken in the original Hilbert space where the constraints are not fulfilled.

By definition, the coherent states span the original, or kinematical, Hilbert space \mathfrak{H} , and thus we can characterize the coherent states themselves by giving their overlap with an arbitrary coherent state. In so doing, we choose the fiducial vector $|\eta\rangle$ so that the overlap is given by

$$\begin{aligned} \langle \pi'', \gamma'' | \pi', \gamma' \rangle &= \exp \left[-2 \int b(x) d^3x \right. \\ &\times \ln \left(\frac{\det \{ \frac{1}{2} [g''^{ab}(x) + g'^{ab}(x)] + \frac{1}{2} i b(x)^{-1} [\pi''^{ab}(x) - \pi'^{ab}(x)] \}}{\{ \det [g''^{ab}(x)] \det [g'^{ab}(x)] \}^{1/2}} \right) \left. \right] , \quad (10) \end{aligned}$$

where $b(x)$, $0 < b(x) < \infty$, is a scalar density which is discussed below.

Additionally, we observe that γ'' and γ' do *not* appear in the explicit functional form given in (10). In particular, the smooth matrix γ has been replaced by the smooth matrix g which is defined at every point by

$$g(x) \equiv e^{\gamma(x)/2} \tilde{g}(x) e^{\gamma(x)^T/2} \equiv \{g_{ab}(x)\} , \quad (11)$$

where the matrix $\tilde{g}(x) \equiv \{\tilde{g}_{ab}(x)\}$ is given by (7). The map $\gamma \rightarrow g$ is clearly many-to-one since γ has nine independent variables at each point while g , which is symmetric, has only six. In view of this functional dependence we may denote the given functional in (10) by $\langle \pi'', g'' | \pi', g' \rangle$, and henceforth we shall adopt this notation. In particular, we note that (8) and (9) become

$$\langle \pi, g | \hat{g}_{ab}(x) | \pi, g \rangle \equiv g_{ab}(x) , \quad (12)$$

$$\langle \pi, g | \hat{\pi}_c^a(x) | \pi, g \rangle = \pi^{ab}(x) g_{bc}(x) \equiv \pi_c^a(x) , \quad (13)$$

which show that the meaning of the labels π and g is that of *mean* values rather than sharp eigenvalues.

Reproducing kernel Hilbert spaces

Although not commonly used, reproducing kernel Hilbert spaces are very natural and readily understood. By definition, the vectors $\{|\pi, g\rangle\}$ span the Hilbert space \mathfrak{H} , and therefore two elements of a dense set of vectors have the form

$$|\phi\rangle = \sum_{j=1}^J \alpha_j |\pi_{[j]}, g_{[j]}\rangle , \quad (14)$$

$$|\psi\rangle = \sum_{k=1}^K \beta_k |\pi_{(k)}, g_{(k)}\rangle , \quad (15)$$

for general sets $\{\alpha_j\}_{j=1}^J$, $\{\beta_k\}_{k=1}^K$, $\{\pi_{[j]}, g_{[j]}\}_{j=1}^J$, $\{\pi_{(k)}, g_{(k)}\}_{k=1}^K$, and some $J, K < \infty$. The inner product of two such vectors is clearly given by

$$\langle \phi | \psi \rangle = \sum_{j,k=1}^{J,K} \alpha_j^* \beta_k \langle \pi_{[j]}, g_{[j]} | \pi_{(k)}, g_{(k)} \rangle . \quad (16)$$

To *represent* the abstract vectors themselves as functionals, we adopt the natural coherent-state representation, i.e.,

$$\begin{aligned}\phi(\pi, g) &\equiv \langle \pi, g | \phi \rangle = \sum_{j=1}^J \alpha_j \langle \pi, g | \pi_{[j]}, g_{[j]} \rangle , \\ \psi(\pi, g) &\equiv \langle \pi, g | \psi \rangle = \sum_{k=1}^K \beta_k \langle \pi, g | \pi_{(k)}, g_{(k)} \rangle .\end{aligned}\tag{17}$$

Thus, we have a dense set of continuous functions and a definition of an inner product between pairs of such functions given by

$$(\phi, \psi) \equiv \langle \phi | \psi \rangle ,\tag{18}$$

as defined in (16). It only remains to complete the space to a (separable) Hilbert space \mathfrak{C} , composed entirely of continuous functions, by adding the limit points of all Cauchy sequences in the norm $\|\psi\| \equiv (\psi, \psi)^{1/2}$. Note well that *all properties* of the reproducing kernel Hilbert space \mathfrak{C} follow as direct consequences from the continuous coherent-state overlap function $\langle \pi'', g'' | \pi', g' \rangle$ itself; for details see, e.g., [6].

3 Projection Operator Approach Toward Quantum Constraints

Consider a classical phase space system with a set of constraints given by $\phi_\alpha(p, q) = 0$ for all α , $1 \leq \alpha \leq A$, which defines the constraint hypersurface $\mathcal{C} \equiv \{(p, q) : \phi_\alpha(p, q) = 0, \text{ for all } \alpha\}$. Such constraints are added to the classical Hamiltonian $H(p, q)$ with the help of Lagrange multipliers $\{\lambda^\alpha(t)\}$ to form the total Hamiltonian

$$H_T(p, q) = H(p, q) + \lambda^\alpha \phi_\alpha(p, q) .\tag{19}$$

The time derivative of the constraints must vanish as well, and this condition leads to

$$\dot{\phi}_\alpha = \{\phi_\alpha, H\} + \lambda^\beta \{\phi_\alpha, \phi_\beta\} = 0 .\tag{20}$$

First class constraints arise when both Poisson brackets vanish on \mathcal{C} , and therefore

$$\{\phi_\alpha, \phi_\beta\} = c_{\alpha\beta}{}^\gamma \phi_\gamma , \quad (21)$$

$$\{\phi_\alpha, H\} = h_\alpha{}^\beta \phi_\beta . \quad (22)$$

If $c_{\alpha\beta}{}^\gamma$ are constants, then the system is called closed first class; if instead $c_{\alpha\beta}{}^\gamma$ are functions of the phase space variables, then the system is called open first class. In either case, the Lagrange multipliers are not determined by the equations of motion and must be chosen (a “gauge” choice) to find the solution of the equations of motion.

Instead, if the Poisson bracket of the constraints does *not* vanish on \mathcal{C} , assuming for illustration that it has an inverse, then it follows that

$$\lambda^\beta \equiv -\{\phi_\alpha, \phi_\beta\}^{-1} \{\phi_\alpha, H\} , \quad (23)$$

which means that the Lagrange multipliers are determined by the equations of motion. In this case, the constraints are referred to as second class. Of course, there can be intermediate cases for which some of the constraints are first class while the remainder are second class.

The Dirac approach to the quantization of constraints requires quantization before reduction. Thus the constraints are first promoted to self-adjoint operators,

$$\phi_\alpha(p, q) \rightarrow \Phi_\alpha(P, Q) , \quad (24)$$

for all α , and then the physical Hilbert space \mathfrak{H}_{phys} is defined by those vectors $|\psi\rangle_{phys}$ for which

$$\Phi_\alpha(P, Q)|\psi\rangle_{phys} = 0 \quad (25)$$

for all α . This procedure works for a limited set of classical first class constraint systems, but it does not work in general and especially not for second class constraints.

The projection operator approach to quantum constraints proceeds by offering a slight generalization of the Dirac procedure. Instead of insisting that (25) holds exactly, we introduce a projection operator \mathbb{E} defined by

$$\mathbb{E} = \mathbb{E}(\Sigma_\alpha \Phi_\alpha^2 \leq \delta(\hbar)^2) , \quad (26)$$

where $\delta(\hbar)$ is a positive *regularization parameter* and we have assumed that $\Sigma_\alpha \Phi_\alpha^2$ is self adjoint. This relation means that \mathbb{E} projects onto the spectral range of the self-adjoint operator $\Sigma_\alpha \Phi_\alpha^2$ in the interval $[0, \delta(\hbar)^2]$. In this case, $\mathfrak{H}_{phys} = \mathbb{E}\mathfrak{H}$. As a final step, the parameter $\delta(\hbar)$ is reduced as much as required, and, in particular, when some second-class constraints are involved, $\delta(\hbar)$ ultimately remains strictly positive. This general procedure treats all constraints simultaneously and treats them all on an equal basis; see [7].

Several examples illustrate how the projection operator method works. If $\Sigma_\alpha \Phi_\alpha^2 = J_1^2 + J_2^2 + J_3^2$, the Casimir operator of $su(2)$, then $0 \leq \delta(\hbar)^2 < 3\hbar^2/4$ works for this first class example. If $\Sigma_\alpha \Phi_\alpha^2 = P^2 + Q^2$, where $[Q, P] = i\hbar\mathbb{1}$, then $\hbar \leq \delta(\hbar)^2 < 3\hbar$ covers this second class example. If the single constraint $\Phi = Q$, an operator whose zero lies in the continuous spectrum, then it is convenient to take an appropriate form limit of the projection operator as $\delta \rightarrow 0$; see [7]. The projection operator scheme can also deal with irregular constraints such as $\Phi = Q^3$, and even mixed examples with regular and irregular constraints such as $\Phi = Q^3(1 - Q)$, etc.; see [8].

It is also of interest that the desired projection operator has a general, time-ordered integral representation (see [9]) given by

$$\mathbb{E} = \mathbb{E}(\Sigma_\alpha \Phi_\alpha^2 \leq \delta(\hbar)^2) = \int \mathbb{T} e^{-i \int \lambda^\alpha(t) \Phi_\alpha dt} \mathcal{D}R(\lambda) . \quad (27)$$

The weak measure R depends on the number of Lagrange multipliers, the time interval, and the regularization parameter $\delta(\hbar)^2$. The measure R does *not* depend on the constraint operators, and thus this relation is an operator identity, holding for any set of operators $\{\Phi_\alpha\}$. The time-ordered integral representation for \mathbb{E} given in (27) can be used in path-integral representations as will become clear below.

4 Continuous-time Regularized Functional Integral Representation without/with Constraints

It is useful to reexpress the coherent-state overlap function by means of a functional integral. This process can be aided by the fact that the expression (10) is analytic in the variable $g''^{ab}(x) + ib(x)^{-1}\pi''^{ab}(x)$ up to a factor. As

a consequence the elements of the reproducing kernel Hilbert space satisfy a complex polarization condition, which leads to a second-order differential operator that annihilates each element of \mathfrak{C} . This fact can be used to generate a functional representation of the form

$$\begin{aligned}
\langle \pi'', g'' | \pi', g' \rangle &= \exp \left[-2 \int b(x) d^3x \right. \\
&\quad \times \ln \left(\frac{\det \{ \frac{1}{2} [g''^{ab}(x) + g'^{ab}(x)] + \frac{1}{2} i b(x)^{-1} [\pi''^{ab}(x) - \pi'^{ab}(x)] \}}{\{ \det [g''^{ab}(x)] \det [g'^{ab}(x)] \}^{1/2}} \right) \Big] \\
&= \lim_{\nu \rightarrow \infty} \overline{\mathcal{N}}_\nu \int \exp[-i \int g_{ab} \dot{\pi}^{ab} d^3x dt] \\
&\quad \times \exp \{ -(1/2\nu) \int [b(x)^{-1} g_{ab} g_{cd} \dot{\pi}^{bc} \dot{\pi}^{da} + b(x) g^{ab} g^{cd} \dot{g}_{bc} \dot{g}_{da}] d^3x dt \} \\
&\quad \times [\Pi_{x,t} \Pi_{a \leq b} d\pi^{ab}(x, t) dg_{ab}(x, t)] . \tag{28}
\end{aligned}$$

Here, because of the way the new independent variable t appears in the right-hand term of this equation, it is natural to interpret t , $0 \leq t \leq T$, $T > 0$ as coordinate “time”. The fields on the right-hand side all depend on space and time, i.e., $g_{ab} = g_{ab}(x, t)$, $\dot{g}_{ab} = \partial g_{ab}(x, t)/\partial t$, etc., and, importantly, the integration domain of the formal measure is strictly limited to the domain where $\{g_{ab}(x, t)\}$ is a positive-definite matrix for all x and t . For the boundary conditions, we have $\pi'^{ab}(x) \equiv \pi^{ab}(x, 0)$, $g'_{ab}(x) \equiv g_{ab}(x, 0)$, as well as $\pi''^{ab}(x) \equiv \pi^{ab}(x, T)$, $g''_{ab}(x) \equiv g_{ab}(x, T)$ for all x . Observe that the right-hand term holds for any T , $0 < T < \infty$, while the left-hand and middle terms are independent of T altogether.

In like manner, we can incorporate the constraints into a functional integral by using an appropriate form of the integral representation (27). The resultant expression has a functional integral representation given by

$$\begin{aligned}
\langle \pi'', g'' | \mathbb{E} | \pi', g' \rangle &= \int \langle \pi'', g'' | \mathbf{T} e^{-i \int [N^a \mathcal{H}_a + N \mathcal{H}] d^3x dt} | \pi', g' \rangle \mathcal{D}R(N^a, N) \\
&= \lim_{\nu \rightarrow \infty} \overline{\mathcal{N}}_\nu \int e^{-i \int [g_{ab} \dot{\pi}^{ab} + N^a H_a + N H] d^3x dt} \\
&\quad \times \exp \{ -(1/2\nu) \int [b(x)^{-1} g_{ab} g_{cd} \dot{\pi}^{bc} \dot{\pi}^{da} + b(x) g^{ab} g^{cd} \dot{g}_{bc} \dot{g}_{da}] d^3x dt \} \\
&\quad \times [\Pi_{x,t} \Pi_{a \leq b} d\pi^{ab}(x, t) dg_{ab}(x, t)] \mathcal{D}R(N^a, N) . \tag{29}
\end{aligned}$$

Despite the general appearance of (29), we emphasize once again that this representation has been based on the affine commutation relations and *not* on any canonical commutation relations.

The expression $\langle \pi'', g'' | \mathbb{E} | \pi', g' \rangle$ denotes the coherent-state matrix elements of the projection operator \mathbb{E} which projects onto a subspace of the original Hilbert space on which the quantum constraints are fulfilled in a regularized fashion. Furthermore, the expression $\langle \pi'', g'' | \mathbb{E} | \pi', g' \rangle$ is another continuous functional that can be used as a reproducing kernel and thus used directly to generate the reproducing kernel physical Hilbert space on which the quantum constraints are fulfilled in a regularized manner. Up to a surface term, the phase factor in the functional integral represents the canonical action for general relativity, and specifically, N^a and N denote Lagrange multiplier fields (classically interpreted as the shift and lapse), while H_a and H denote phase-space symbols (since $\hbar \neq 0$) associated with the quantum diffeomorphism and Hamiltonian constraint field operators, respectively.

The “diagonal representation”

It is noteworthy that the connection between the Hamiltonian constraint operator field $\mathcal{H}(x)$ and its associated symbol $H(x)$ that is used in the functional integral (29) is closely related to the “diagonal representation” that Sudarshan introduced into quantum optics [10]. In particular,

$$\mathcal{H}(x) = \overline{\mathcal{N}} \int H(x) |\pi, g\rangle \langle \pi, g| [\Pi_x \Pi_{a \leq b} d\pi^{ab}(x) dg_{ab}(x)] . \quad (30)$$

A similar relation connects $\mathcal{H}_a(x)$ to its symbol $H_a(x)$ for all a , $1 \leq a \leq 3$.

Properties of the regularization

The ν -dependent factor in the integrand of (28) and (29) formally tends to unity in the limit $\nu \rightarrow \infty$; but prior to that limit, the given expression regularizes and essentially gives genuine meaning to the heuristic, formal functional integral that would otherwise arise if such a factor were missing altogether [4]. The given form, and in particular the nondynamical, nonvanishing, arbitrarily chosen scalar density $b(x)$, is very welcome since this form leads to a reproducing kernel Hilbert space for gravity having the needed infinite dimensionality; a seemingly natural alternative [11] using $\sqrt{\det[g_{ab}(x)]}$ in place of $b(x)$ fails to lead to a reproducing kernel Hilbert space with the required dimensionality [12]. The choice of $b(x)$ determines a specific ultralocal representation for the basic affine field variables, but this unphysical

and temporary representation disappears entirely after the gravitational constraints are fully enforced (as soluble examples explicitly demonstrate [5]). The integration over the Lagrange multiplier fields (N^a and N) involves a specific measure $R(N^a, N)$, which is normalized such that $\int \mathcal{D}R(N^a, N) = 1$. This measure is designed to enforce (a regularized version of) the quantum constraints; it is manifestly **not** chosen to enforce the classical constraints, even in a regularized form. The consequences of this choice are *profound* in that no (dynamical) gauge fixing is needed, no ghosts are required, no Dirac brackets are necessary, etc. In short, no auxiliary structure of any kind is introduced.

The gravitational anomaly

The quantum gravitational constraints, $\mathcal{H}_a(x)$, $1 \leq a \leq 3$, and $\mathcal{H}(x)$, formally satisfy the commutation relations

$$\begin{aligned} [\mathcal{H}_a(x), \mathcal{H}_b(y)] &= i\frac{1}{2}[\delta_{,a}(x, y)\mathcal{H}_b(y) + \delta_{,b}(x, y)\mathcal{H}_a(x)] , \\ [\mathcal{H}_a(x), \mathcal{H}(y)] &= i\delta_{,a}(x, y)\mathcal{H}(y) , \\ [\mathcal{H}(x), \mathcal{H}(y)] &= i\frac{1}{4}\delta_{,a}(x, y)[g^{ab}(x)\mathcal{H}_b(x) + \mathcal{H}_b(x)g^{ab}(x) \\ &\quad + g^{ab}(y)\mathcal{H}_b(y) + \mathcal{H}_b(y)g^{ab}(y)] . \end{aligned} \tag{31}$$

Following Dirac, we first suppose that $\mathcal{H}_a(x)|\psi\rangle_{phys} = 0$ and $\mathcal{H}(x)|\psi\rangle_{phys} = 0$ for all x and a , where $|\psi\rangle_{phys}$ denotes a vector in the physical Hilbert space \mathfrak{H}_{phys} . However, these conditions are *incompatible* since $[\mathcal{H}_b(x), g^{ab}(x)] \neq 0$ and almost surely $g^{ab}(x)|\psi\rangle_{phys} \notin \mathfrak{H}_{phys}$, even when smeared. As noted previously, this means that the quantum gravitational constraints are partially second class.

5 Hard-core Picture of Nonrenormalizability

Nonrenormalizable quantum field theories involve an infinite number of distinct counterterms when approached by a regularized, renormalized perturbation analysis. Focusing on scalar field theories, a qualitative Euclidean functional integral formulation is given by

$$S_\lambda(h) = \mathcal{N}_\lambda \int e^{\int h\phi d^n x - W_o(\phi) - \lambda V(\phi)} \mathcal{D}\phi , \tag{32}$$

where $W_o(\phi) \geq 0$ denotes the free action and $V(\phi) \geq 0$ the interaction term. If $\lambda = 0$, the support of the integral is determined by $W_o(\phi)$; when $\lambda > 0$, the support is determined by $W_o(\phi) + \lambda V(\phi)$. Formally, as $\lambda \rightarrow 0$, $S_\lambda(h) \rightarrow S_0(h)$, the functional integral for the free theory. However, it may happen that

$$\lim_{\lambda \rightarrow 0} S_\lambda(h) = S'_0(h) \neq S_0(h) , \quad (33)$$

where $S'_0(h)$ defines a so-called *pseudofree* theory. Such behavior arises formally if $V(\phi)$ acts as a hard core, projecting out certain fields that are not restored to the support of the free theory as $\lambda \rightarrow 0$ [13]. In particular, for relativistic φ_n^4 models, it is known [14], provided $\phi \not\equiv 0$, that

$$\frac{[\int \phi(x)^4 d^n x]^{1/2}}{\int \{[\nabla \phi(x)]^2 + m^2 \phi(x)^2\} d^n x} \leq \frac{4}{3} , \quad (34)$$

for $n = 3, 4$, while for $n \geq 5$, no finite upper bound exists. Although such inequalities are derived for test functions, the bound on the quotient still applies to the limit in which a sequence of test functions weakly converges to a distribution. Such qualitatively different behavior for $n \leq 4$ and $n \geq 5$ coincides with the division of such models into renormalizable and nonrenormalizable categories. Based on this fact, it is highly suggestive that nonrenormalizable models have support properties that are significantly influenced by the hard-core nature of $V(\phi)$ relative to $W_o(\phi)$, a property that also accounts for the need of an infinite set of distinct perturbative counterterms.

It is noteworthy that there exist highly idealized nonrenormalizable model quantum field theories with exactly the behavior described; see [14]. It is our belief that these soluble models strongly suggest that nonrenormalizable φ_n^4 models can be understood by the same mechanism, and that they too can be properly formulated by the incorporation of a limited number of counterterms distinct from those suggested by a perturbation treatment. Although technically more complicated, we see no fundamental obstacle in dealing with quantum gravity on the basis of an analogous hard-core interpretation. However, that is a problem for the future.

Dedication

I am pleased to dedicate this article to the 75th birthday of George Sudarshan, and I wish him many more years of good health and productive research.

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